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ELECTING LEGISLATURES

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Abstract and Acknowledgement

A "legislature" is defined to be an assembly of at least two elected officials which selects final policy outcomes. Legislative elections therefore concern the electoral choice of such an assembly. The classical two-candidate, single-district, model of electoral competition is not a legislative election in the sense of this essay. In the classical model the legislature comprises the winning candidate: this agent has monopolistic control of the legislative decision-making machinery, and implements his winning policy. With this system, voters have a straightforward "best" voting rule for any pair of candidate positions offered in the election: vote sincerely. In the multi-stage legislative electoral system, final outcomes depend on the entire composition of the legislature and the specifics of legislative decision-making. With such a system, voters' decisions are considerably less straightforward, which in turn complicates candidates' strategic choices. This paper presents a fairly technical review of the spatial-theoretic literature on legislative elections.

The paper was commissioned by Norman Schofield for the conference on *Coalition Theory and Public Choice* (Fiesole, Italy: May 1987). On the one hand, the task was easy: the literature is small and much of it involves my own work. On the other hand, the task was difficult: the literature is small and much of it involves my own work. In any event, I am grateful to Professor Schofield for giving me the opportunity and incentive to raise some issues with which I have long been concerned. He is in no way responsible for any errors or omissions the paper might contain. I feel perfectly free, however, to blame him for the appearance of self-indulgence that the essay surely has.

1. Introduction

Policy outcomes in representative democracies arise out of legislative decision-making, and legislatures consist of more than one elected official. The preferences of policy-oriented voters over possible representatives, therefore, will be induced by their preferences over policy outcomes and the institutional structure of legislative decision-making. This observation is, I believe, fundamental to developing models for understanding the election of legislatures. To assume at the outset that individuals vote solely on the basis of the policy positions of the available candidates, and ignore the legislative implications of their vote, is a misspecification of individual payoffs and of the choice set.

The canonic spatial model of electoral competition presumes a degenerate legislative structure. Two Downsian parties (i.e. candidates) compete, under simple plurality voting, for the right to "represent" a society of n individuals who are undifferentiated except through their preferences over policy outcomes. (In particular, the electorate is not partitioned into districts, each of which electing a representative to some legislature.) With this institutional setting, the party winning the election monopolistically selects the final policy outcome. If there is complete information (e.g. Black, 1958; Downs, 1957: ch.8), then there is no loss in generality in identifying the parties with the policy outcome that they would implement if elected. In this case, voters' induced preferences over parties are trivially defined by their preferences over outcomes. When there is incomplete information (e.g. Banks, 1986; Enelow & Hinich, 1984), voters' induced preferences over parties are defined by the composition of preferences over outcomes and beliefs over what a party would implement if elected. Furthermore, whatever informational structure is assumed, the existence of only two competitors for an office which endows the winner with monopolistic control of the policy outcome is sufficient to yield sincere voting as the only sensible Nash equilibrium strategy.¹

It is a recurrent theme of this essay that the simplicity of the induced preferences described above, and the reasonableness of sincere voting, are peculiar to models of single-district, two-party elections under simple-plurality rule. Although important, such models are generally not sufficient

for analysing the election of legislatures.

The plan of the essay is as follows. Section 2 introduces the basic framework used throughout, and Section 3 explores the implications of the sincere voting assumption for models of legislative elections. In the light of the results of this section, Sections 4 and 5 review the extant (formal) theoretical literature on legislative elections. Since a legislature is understood throughout the essay to consist of at least two elected candidates, the review will not deal with models of multicandidate, single-winner elections (e.g. Cox, 1985, 1987). Section 6 contains a concluding discussion.

2. Basic framework²

2.1 Individuals, preferences, and candidates: Let $N = \{1, \dots, n\}$ be a finite set of voters. The set of feasible policy outcomes, X , is assumed isomorphic to a compact subset of the real line, \mathbf{R} . Typically, we take X to be an interval. Let $U(X)$ be the set of strictly single-peaked preference orderings (i.e. no flat spots) on X . Hereafter, the dependency of U on X will be left implicit. A preference profile for N is a list $\mathbf{u} = (u_i)_{i \in N} \in U^n$, where u_i describes individual i 's preferences on X . For any $u_i \in U$, let $x_i = \operatorname{argmax}_X u_i(\cdot)$: x_i is individual i 's ideal point in X . Individuals' preferences are common knowledge.

Let \mathbf{C} be the set of discrete subsets of X . Any $C = \{y_1, \dots, y_t\} \in \mathbf{C}$ is a set of candidate platforms. Typically, $|C| < \infty$. Unless explicitly stated otherwise, the terms "candidate platform" (generically indexed y_k) and "candidate" (generically indexed k) will be used interchangeably: this should cause no confusion.

For any $i \in N$ and set of candidates $C \in \mathbf{C}$, i 's (pure) voting strategy is a function:

$$\sigma_i : U \times \mathbf{C} \rightarrow \mathbf{C},$$

where $\sigma_i(u_i, C) = y_k$ means individual i , with preferences u_i on X , casts his vote for candidate $y_k \in C$. Let $\sigma(\mathbf{u}, C) = (\sigma_1(u_1, C), \dots, \sigma_n(u_n, C))$ denote an arbitrary list of voter strategies (given the

profile \mathbf{u} and the set C). Where there is no danger of ambiguity, the dependency of σ etc. on \mathbf{u} and C , will be suppressed.

2.2 Legislative election structures: Given a set of candidates $C \in \mathcal{C}$, a legislative election structure for C consists of (1) a rule, ϵ_C , governing which candidates get elected to the legislature as a function of individuals' voting strategies, and (2) a rule, λ_C , describing how the elected candidates arrive at a final policy outcome.

Fix a set of candidates C arbitrarily. An election rule for C is a mapping,

$$\epsilon_C: C^n \rightarrow 2^{C \setminus \emptyset}.$$

Call any set $C^* \in 2^{C \setminus \emptyset}$ determined via an election rule ϵ_C , a legislature. Let $v_k(\sigma) = |\{i \in N \mid \sigma_i = y_k\}|$. Then, for example, under the fixed-standard method studied in Greenberg and Weber (1985), $fs(m)$, the legislature generated by the strategy profile σ is defined by:

$$\epsilon_C^{fs(m)}(\sigma) = \{y_k \in C \mid v_k(\sigma) \geq m, m \in \mathbb{N}\}.$$

Let $\Sigma_C = \{\epsilon_C \mid \epsilon_C: C^n \rightarrow 2^{C \setminus \emptyset}\}$.

A legislative outcome function for C (LOF) is a mapping,

$$\lambda_C: (2^{C \setminus \emptyset}) \times C^n \rightarrow X.$$

For every possible legislature elected from C , the LOF defines the legislative policy outcome. This outcome is not restricted a priori to lie in C . For example, the final outcome may be some weighted average of the elected candidates' platforms. And notice that we allow the LOF to depend on voter strategies as well as on the positions of the elected set of candidates. This, for example, permits successful candidates' vote-shares to matter in legislative decision-making. Of course, the LOF may be constant across $\sigma \in C^n$ for any given $C^* \in 2^{C \setminus \emptyset}$. For example, the median successful candidate rule, λ_C^μ , has this property:

$$\lambda_C^\mu(C^*, \sigma) = \text{median} \{y_k \mid y_k \in C^*\}.$$

Let $\Lambda_C = \{\lambda_C \mid \lambda_C: (2^{C \setminus \emptyset}) \times C^n \rightarrow X\}$.

A legislative election structure for $C \in \mathcal{C}$ is thus an ordered pair $(\epsilon_C, \lambda_C) \in \Sigma_C \times \Lambda_C$.

Typically, we are not interested in all logically possible legislative election structures. In this essay, only structures which are anonymous (do not depend on the names of individual voters or candidates), efficient (do not select alternatives which are wanted by noone), and candidate-independent (are well-defined for any set of candidates) are considered.

Definition: $\epsilon_C \in \Sigma_C$ is nontrivial iff $\exists y_k \in C, \exists \sigma \in C^n$ such that $v_k(\sigma) > 0$ and $y_k \notin \epsilon_C(\sigma)$.

$\epsilon_C \in \Sigma_C$ is E-efficient iff $\forall C \in \mathcal{C}, \forall \sigma \in C^n; y_k \in \epsilon_C(\sigma) \Rightarrow v_k(\sigma) > 0$.

ϵ_C is anonymous iff ϵ_C is symmetric with respect to voters.

ϵ_C is C*-independent iff $\forall C \in \mathcal{C}^*, C \supseteq C^*, \forall (\mathbf{u}, C) \in U^n \times \mathcal{C}^*; \epsilon_C(\sigma) = \epsilon(\sigma)$.

Let $\Sigma_C^* = \{\epsilon_C \mid \epsilon_C \text{ is nontrivial, E-efficient, anonymous}\}$ and, by an abuse of notation, let $\Sigma^* = \{\epsilon_C \in \Sigma_C^* \mid \epsilon_C \text{ is C-independent}\}$.

Definition: $\lambda_C \in \Lambda_C$ is L-efficient iff $\forall C^* \in 2^{C \setminus \emptyset}, \forall \sigma \in C^n; \lambda_C(C^*, \sigma) \in \text{co}(C^*)$, where $\text{co}(H)$ is the convex hull of H (a subset of \mathbb{R}).

$\lambda_C \in \Lambda_C$ is anonymous iff it is symmetric with respect to both candidates and voters.

$\lambda_C \in \Lambda_C$ is C*-independent, $C \supseteq C^*$, iff $\forall C^* \in 2^{C \setminus \emptyset}, \forall (\mathbf{u}, C) \in U^n \times \mathcal{C}^*; \lambda_C(C^*, \sigma) = \lambda(C^*, \sigma)$.

Let $\Lambda_C^* = \{\lambda_C \mid \lambda_C \text{ is L-efficient and anonymous}\}$ and $\Lambda^* = \{\lambda_C \in \Lambda_C^* \mid \lambda_C \text{ is C-independent}\}$.

Definition: An ordered pair (ϵ_C, λ_C) is admissible iff $(\epsilon_C, \lambda_C) \in \Sigma^* \times \Lambda^*$. Let $\Omega = \Sigma^* \times \Lambda^*$, and call Ω the set of admissible legislative election structures.

As remarked above, unless explicitly stated otherwise, this essay deals only with admissible legislative election structures.

2.3 Reduced forms: Given a set of candidates $C \in \mathcal{C}$, an election rule ϵ_C , a LOF λ_C , and a

vector of voting strategies σ , the final legislative outcome is given by $\lambda_C(\epsilon_C(\sigma), \sigma) \in X$. Define the mapping,

$$\gamma_C : C^n \rightarrow X$$

by setting $\gamma_C(\sigma) = \lambda_C(\epsilon_C(\sigma), \sigma)$, $\forall \sigma \in C^n$. If $(\epsilon_C, \lambda_C) \in \Omega$, then γ_C is anonymous (i.e. symmetric with respect to voters), efficient (i.e. $\forall \sigma \in C^n$, $\gamma_C(\sigma) \in \text{co}(C)$), and C -independent (i.e. $\forall (u, C) \in U^n \times C$, $\gamma_C(\sigma(u, C)) = \gamma(\sigma(u, C))$). The mapping γ_C will be referred to as the reduced form of the legislative election structure for C . Let,

$$\Gamma^* = \{\gamma_C : C^n \rightarrow X \mid \gamma_C(\cdot) = \lambda_C(\epsilon_C(\cdot), \cdot), (\epsilon_C, \lambda_C) \in \Omega\}$$

be the set of reduced forms of admissible election structures.

Given any legislative election structure, define i 's indirect utility by,

$$w_i(\sigma(u, C)) \equiv u_i(\gamma_C(\sigma(u, C))), i \in N.$$

Thus w_i describes i 's payoff under the legislative election structure $(\epsilon_C, \lambda_C) \in \Omega$ as a function of the voting strategy vector $\sigma(u, C)$; $w_i : C^n \rightarrow \mathbb{R}$.

This completes the description of the basic framework. Some important concepts are noticeably absent: in particular, candidate objectives and any notion of equilibrium. However, since these vary across models, their introduction is deferred until necessary.

3. The assumption of sincere voting

It was claimed in the Introduction that an assumption of sincere voting is inappropriate for models of legislative elections. This claim is justified on theoretical grounds below, where it is argued that sincere voting in legislative elections constitutes rational behaviour on the part of voters only if the legislative election structure is equivalent to a two-candidate, single-winner competition. Given such circumstances, it is hard to understand the rationale for (nondegenerate) legislative election structures. Furthermore, the assumption is questionable empirically (see Riker, 1982 sect.VI, for a brief review of the relevant literature): even in large electorates, individuals seem to

vote strategically.

For this section, assume all individuals' preferences are Euclidean:

$$\forall i \in N, \forall y, z \in X; u_i(y) > u_i(z) \Leftrightarrow |x_i - y| < |x_i - z|.$$

Let U_e be the set of Euclidean preferences on X . For $u \in U_e^n$, preferences are effectively characterized by ideal points. Hence, we can replace such preference profiles by vectors of ideal points, $x = (x_i)_N \in X^n$. Because of the structure of voting strategies (cast a vote for a single candidate), assuming Euclidean preferences is not critical for the argument to follow.³

Fix a set of candidates C in X arbitrarily, and let the legislative election structure be given.

Definition: Individual $i \in N$ votes sincerely with respect to C iff:

$$\sigma_i(x_i, C) = y_k \Rightarrow \sim [\exists y_k' \in C \setminus \{y_k\} \mid u_i(y_k') > u_i(y_k)].$$

Let $\sigma_i^T(x_i, C)$ denote i 's sincere voting strategy with respect to C .

An assumption that individuals necessarily vote sincerely with respect to C (for any C) is frequently invoked in models of legislative elections (e.g. Sugden, 1984; Greenberg and Weber, 1985; Greenberg and Shepsle, 1987). If voters are presumed to be rational and outcome-oriented, then this assumption is tantamount to claiming that sincere voting is a weakly dominant strategy.

Definition: The sincere voting strategy σ_i^T is weakly dominant under γ_C for i iff:

$$w_i(\sigma_i^T, \sigma_{-i}) \geq w_i(\sigma_i, \sigma_{-i}), \forall \sigma_i \neq \sigma_i^T, \forall \sigma_{-i},$$

where $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$.

Suppose that sincere voting is invariably a weakly dominant strategy for all individuals. What does this imply for the legislative election structure? To answer this question, we need one further concept.

Definition: A legislative election structure (ϵ_C, λ_C) is straightforward iff:

$$\forall \mathbf{x} \in X^n, \forall i \in N; \sigma_i^T(x_i, C) \text{ is weakly dominant under the reduced form } \gamma_C.$$

When (ϵ_C, λ_C) is straightforward, we say also that γ_C is straightforward.

Theorem 1 [Austen-Smith, 1987b]: No admissible legislative election structure is straightforward.

Essentially, this result is a corollary of a theorem due to Moulin (1980).

There are several ways to understand the result. Perhaps the most immediate is to consider an arbitrary legislative election structure, $(\epsilon_C', \lambda_C') \in \Sigma_C^* \times \Lambda_C^*$ with reduced form γ_C' . Then the following is true:

Theorem 2 [Austen-Smith, 1987b]: $[\gamma_C'$ is C-independent and straightforward] $\Leftrightarrow \exists$ an order statistic ρ on N such that, $\forall (\mathbf{x}, C) \in X^n \times C$,

$$u_{i(\rho)}(y_k) > u_{i(\rho)}(y_{k'}), \forall y_{k'} \in C \setminus \{y_k\} \Rightarrow y_k = \gamma_C'(\sigma(\mathbf{x}, C)),$$

where $i(\rho) \in N$ is the individual with the ρ^{th} -ranked ideal point.

Theorem 2 says that if an assumption of sincere voting at the election stage in any legislative election model is justified on rationality grounds, then which candidates actually get elected to office is immaterial for legislative policy. It is the entire set of candidates competing for office that matters. Moreover, there exists an individual i^* such that if some candidate, c , adopts i^* 's ideal point in the election, x_{i^*} , then (given that sincere voting constitutes rational behaviour) x_{i^*} must be the final legislative outcome, whether or not c is elected to the legislature. So, for example, bargaining in legislatures to reach a compromise policy position is generally incompatible with sincere voting at the election stage. In effect, the multistage structure of a legislative election is irrelevant: whenever

sincere voting is rational, a two-candidate race under an appropriately chosen "q-rule" (i.e. a rule by which the candidate with at least q votes wins outright) generates precisely the same set of policy outcomes. An example illustrates these points.

Example 1: Suppose there are seven voters and the policy space is $[0, 1]$. Let the ideal points be, $\mathbf{x} = (0, 1/5, 1/5, 5/7, 4/5, 4/5, 1)$. Suppose there are three candidates competing for seats in a 2-member legislature, and suppose the election rule is the fixed-standard scheme (ϵ_C^{fs}) with $m = 3$. Let the set of candidate platforms be $C = \{1/5, 5/7, 4/5\}$. Under sincere voting, individuals 1, 2 and 3 vote for 1/5, individual 4 votes for 5/7, and individuals 5, 6 and 7 vote for 4/5. Therefore, the legislature is $C^* = \{1/5, 4/5\}$. How is the final legislative outcome selected? Suppose the LOF is: if either elected candidate wins an overall majority of votes, then that candidate implements his electoral platform; otherwise the final outcome is a weighted sum of the elected candidates' platforms, with the weights being given by their respective vote-shares.⁴ This LOF is certainly anonymous, L-efficient and C-independent. Under sincere voting, the final outcome is 3/7. But if individual 4 votes strategically for 4/5, given the others vote sincerely, the final outcome is 4/5. And this improves 4's payoff over that achieved by sincere voting. To support sincere voting here, the final outcome must be individual 4's ideal point. But then two-candidate simple majority voting generates the same outcome as the two-stage legislative election scheme.

The results reported above have two immediate implications for models of legislative elections. First, if the multi-stage legislative election structure is not vacuous (under complete information), then the assumption of sincere voting is problematic. Individual voting behaviour should be deduced, not presumed. Second, to understand strategic behaviour by candidates and (policy-oriented) voters at the election stage, the structure of the subsequent legislative stage needs to be specified explicitly. Different legislative structures induce different electoral behaviour: if results

on the election stage are to have content, the legislative stage cannot be treated as a "black box".

4. Single district models

In general, parties and candidates are distinct: candidates for office may or may not be members of some party, and parties rarely consist solely of electoral candidates. In single district models, however, parties are typically assumed to be characterized by a single candidate for office (but see Aldrich, 1983a, 1983b). Consequently, in this section, the terms candidate and party will be used interchangeably. Later, the two notions will be distinguished.

Greenberg and Weber (1985) examine a model of proportional representation under the fixed-standard election rule (fs(m)):

$$\epsilon_C^{fs(m)}(\sigma) = \{y_k \in C \mid v_k(\sigma) \geq m, m \in \mathbb{N}\}.$$

Under this rule, the size of the legislature is variable and comprises all those candidates who receive at least m votes in the election, where $m > 0$ is some prespecified integer no greater than n , the size of the electorate. The feasible set X is assumed finite, and individuals' preferences -- although single-peaked -- are not necessarily symmetric on X . In the model, candidates are constrained to adopt distinct positions. Therefore, the feasible sets of candidates, C , is the power set of X . The focus of attention is on the existence and structure of fs(m)-equilibria:

Definition: $C \in \mathcal{C}$ is an fs(m)-equilibrium if, $\forall y_k \in X, v_k(\sigma^T(u, C)) \geq m$ iff $y_k \in C$.

Thus, an fs(m)-equilibrium is "a set of alternatives such that when the voters are faced with the choice among these alternatives, and when each voter votes for the alternative he prefers best in this set, each alternative receives at least m votes and, moreover, no new (potential) candidate can attract m voters by offering another alternative, in addition to those offered in the m-equilibrium" (Greenberg and Weber, 1985, p.696). The main result is:

Theorem 3 [Greenberg & Weber, 1985]: For any $m, 0 < m \leq n$, there exists an fs(m)-equilibrium.

This result, which will be considered in more detail shortly, stands in contrast to that of Greenberg and Shepsle (1987), who study the fixed-number election rule in an otherwise identical environment as that of Greenberg and Weber. The fixed-number election rule (fn(K)) is defined by:

$$\epsilon_C^{fn(K)}(\sigma) = \{y_k \in C \mid v_k(\sigma) < v_{k'}(\sigma) \text{ for at most } K-1 \text{ candidates } y_{k'} \in C \setminus \{y_k\}, K \in \mathbb{N}\}.$$

(If there are ties, it is assumed throughout that a fair random device casts a deciding "vote".) Under the fixed-standard rule, the size of the legislature is variable. Under the fixed-number rule, however, the size of the legislature is predetermined at $K, 0 < K \leq n$.

Definition: $C \in \mathcal{C}$ is an fn(K)-equilibrium if:

- (1) $|C| = K$,
- (2) $\forall y_k \in X \setminus C, v_k(\sigma^T(u, C \cup \{y_k\})) \leq v_{k^*}(\sigma^T(u, C \cup \{y_k\})), \forall y_{k^*} \in C$.

An fn(K)-equilibrium is therefore a set C of exactly K candidates, each adopting a distinct electoral platform such that no additional candidate could enter the election and gain more (sincere) votes as any candidate in C .

Theorem 4 [Greenberg & Shepsle, 1987]: $\forall K \geq 2, \exists u \in U^n$ such that there exists no fn(K)-equilibrium.

At first glance, Theorems 3 and 4 suggest that proportional representation systems using a fixed-standard election rule are likely to be more stable -- at least, at the electoral level -- than those using a fixed-number rule. However, in the absence of a more complete specification of the

legislative election game, this conclusion is premature. As they stand, the results refer to the abstract properties of particular preference aggregation mechanisms. For example, Theorem 3 answers the following question: Given an arbitrary $u \in U^n$, and given an integer m , $0 < m \leq n$, does there exist a subset C of X such that, $\forall y_k \in C$, $|\{i \in N : u_i(y_k) > u_i(y_{k'}), \forall y_{k'} \in C \setminus \{y_k\}\}| \geq m$ and \exists no $y_j \in X \setminus C$ such that $|\{i \in N : u_i(y_j) > u_i(y_k), \forall y_k \in C\}| \geq m$? Although the answer to this question (and the analogous one answered by Theorem 4) is important for the study of proportional representation, it does not, per se, address issues of electoral equilibrium. The difficulty is that the concepts of $fs(m)$ -equilibrium and $fn(K)$ -equilibrium are not behavioural. Substantive conclusions drawn from Theorems 3 and 4, therefore, must be treated cautiously.⁵

For a model of elections, voters and candidates need to be endowed (at least) with strategy sets. In Greenberg/Weber and Greenberg/Shepsle, voters' strategy sets are degenerate, and candidates for office are presumed to seek sufficient "votes" to win office: these restrictions are implicit in the formal definition of $fn(K)$ -equilibrium and $fs(m)$ -equilibrium.⁶ Given these restrictions, the structure of the games for which $fn(K)$ - and $fs(m)$ -equilibria might define solutions does not seem particularly well-suited to legislative elections. To see this, recall Example 1 (Section 3). In that environment, the set $C = \{1/5, 4/5\}$ is the unique $fs(3)$ -equilibrium and the unique $fn(2)$ -equilibrium. However, if candidates care about influencing policy and if sincere voting is rational, then a new candidate could enter at $5/7$ and induce $5/7$ as the legislative policy choice. This entrant would surely not get elected but, by the arguments of section 3, for sincere voting to be rational the implicit legislative outcome function must nevertheless select $5/7$ as the final policy.⁷ Consequently, for sincere voting to be rational and for $fs(m)$ - and $fn(K)$ -equilibria to be germane for legislative elections, candidates must be presumed to seek office for reasons other than affecting policy. That there are nonpolicy reasons for getting elected is unexceptional; that such reasons can be the only ones is problematic. If elected candidates do in fact affect policy in the legislature then, by Theorem 1, sincere voting and rational voters cannot be jointly assumed: the legislative outcome function needs to be made explicit and appropriate voter (and candidate) behaviour deduced. It is

not sufficient to specify the election rule alone and impose sincere voting over candidates as a behavioural rule for voters.

A model similar in spirit to those of Greenberg et.al. is due to Robert Sugden (1984). He examines a normative principle of Free Association "as a game of strategy in which voters bargain and coalesce with one another in an attempt to influence the composition of the assembly" (p.33). The principle, deriving from arguments of J.S.Mill (1861) and Carl Andrae (1926) among others, is that "citizens should be free to choose for themselves which constituencies they belong to" (Sugden, *ibid.*, p.32). The analytical issue addressed is the extent to which particular forms of proportional representation implement this normative principle. His approach is ingenious. Sugden first describes a cooperative game without sidepayments in which individual voters can freely collude to elect some set of candidates. Then his "criterion for evaluating schemes of proportional representation is that, so long as the core of that game is not empty, the set of candidates who are elected [under the scheme] should be a member of the core" (p.40). He concludes that under single-peaked preferences, a modified version of Single-Transferable Voting (STV) satisfies this criterion.

Unfortunately, when evaluating outcomes under STV, he assumes sincere voting over candidates, ignoring the legislative policy implications of such behaviour. Similarly, in the game-theoretic model used as a benchmark -- which is of special interest here -- individuals are concerned about electing slates of candidates, and preferences over slates are induced without concern for the final policy outcomes generated from any elected legislature.

Sugden assumes a fixed number of candidates, t : hence, $C \in C_t = \{C \in C \mid |C| = t\}$. The election is by the fixed-standard rule, $fs(m)$. The standard m is set so that at most K candidates can be elected, and m is the Droop quota -- the smallest integer exceeding $n/(K+1)$. Assume that $t > K$, and suppose (for convenience) that $(n+1)/(K+1) = m$ is integer. Then define a slate to be any subset A of C such that $|A| \leq K$. Preferences over slates are derived from the underlying (strict) preferences

over candidates by: Slate A is preferred to slate B by individual i iff either (i) $\text{argmax}_{A \setminus (A \cap B)} u_i > \text{argmax}_{B \setminus (A \cap B)} u_i$, or (ii) $A \supset B$. If i prefers A to B according to this criterion, write $A \succ_i B$, etc.. In the model, voters are allowed to abstain. But since this plays no role in the analysis, I assume here that everyone always votes. Thus, voter strategies are as specified in section 2, above.

Definition: Fix $C \in \mathcal{C}_t$. A slate A in C is blocked under fs(m) iff $\exists L, N \supseteq L$, and $\exists \sigma_L = (\sigma_i)_{i \in L}$ such that,

$$\forall \sigma_{-L} = (\sigma_j)_{j \notin L}, \forall i \in L, \varepsilon_C^{\text{fs}(m)}((\sigma_L, \sigma_{-L})) \succ_i A.$$

The core(C) under fs(m) is the set of all slates in C that are not blocked under fs(m).

Thus, only the voting game is considered. As in the models of Greenberg et al., candidate objectives are left more-or-less implicit: to obtain sufficient votes to insure election. (And note that core(C) is a beta core.)

Recalling that individual preferences over X are single-peaked, rank-order ideal points so that $x_i < x_{i+1}$ for all $i < n$. For any $C \in \mathcal{C}_t$, let $C^{\text{dq}} = \{y_1, \dots, y_K\}$ be the subset of C such that (1) $y_i = \text{argmax}_{C^{\text{dq}}} u_{im}(\cdot)$, $\forall y_i \in C^{\text{dq}}$; (2) $y_k \neq y_j$, $\forall y_k, y_j \in C^{\text{dq}}$; (3) K = maximum number of seats in the legislature; and (4) m = Droop quota. Since preferences are assumed strict (an implicit restriction on C), C^{dq} is uniquely defined for every C.

Theorem 5 [Sugden, 1984]: $\forall C \in \mathcal{C}_t$, the core(C) under fs(m), where m is the Droop quota, is exactly C^{dq} .

It is instructive to compare Theorem 5 to Theorem 3. Theorem 3 guarantees the existence of at least one (and maybe several) fs(m)-equilibrium for any m -- a fortiori, for the Droop quota. Theorem 5 asserts that under strategic and cooperative voting, there exists a unique equilibrium

(core) set of candidates under fs(m). Where do they differ? The answer is straightforward.

Sugden's equilibrium is defined for any arbitrarily fixed set of distinct candidate platforms: core(C) is a cooperative voting equilibrium concept relative to a given $C \in \mathcal{C}_t$. In contrast, Greenberg and Weber fix voting strategies: by definition of an fs(m) equilibrium, voters (noncooperatively) vote sincerely, and so the focus of the fs(m)-equilibrium concept is candidate platforms relative to σ^T . Therefore, not every fs(m)-equilibrium (m = Droop quota) is necessarily a core slate, and not every core slate will be an fs(m)-equilibrium. However, consider the slate $C^{\text{dq}*} = \{x_{1m}, \dots, x_{Km}\}$, where m and K are defined as for Theorem 5, and x_{jm} is individual jm 's ideal point (with $x_i < x_{i+1}$ for all $i < n$). Then it is simple to check that $C^{\text{dq}*}$ is both an fs(m)-equilibrium and a core(C) slate for any $C \supseteq C^{\text{dq}*}$.

Theorem 5 is driven by the assumption that preferences over slates are given by the induced relation, \succ_i . The assumption is very restrictive. Sugden recognizes this but argues that if a legislature is considered a forum for debate, and not a decision-making body, then it is a legitimate starting point from which to analyse proportional representation. However, legislatures, whatever else they might be, are decision-making bodies, and debate is simply one mechanism for influencing final policy decisions. From this perspective, it is natural to specify a legislative outcome function (which may well involve information transmission through debate) and induce preferences through this. And, once again, there is no reason a priori to suppose such preferences will be sincere.

Despite the objections, Sugden's approach through cooperative theory is important. Most democratic polities involve party activists; agents who, among other things, devote resources to organizing voting blocs. In legislative elections where the composition of the parliament as a whole matters for final outcomes, the ability to coordinate voting strategies among groups of voters is strategically valuable. Party activists promote such coordination, and one possible way to model such activist behaviour is implicitly via a cooperative game at the electoral stage of the process. Of course, any set of voter strategies should be self-enforcing, and, therefore, an appropriate

(cooperative) solution concept must subsume (noncooperative) Nash behaviour. So, for example, although the core is the natural solution concept for the normative issue with which Sugden is concerned, some form of Strong Nash equilibrium might be more appropriate for positive theory.

The importance of being able to coordinate voting strategies in legislative elections with many parties was recognized clearly by Anthony Downs (1957). In chapter 9 of *An Economic Theory of Democracy*, Downs considers a two-stage model in which a multi-party legislature is elected via proportional representation (from a single district), and the elected representatives then use majority voting to determine a government. "Under these conditions, each voter's ballot does not support the policies of any one party. Instead it supports the whole coalition that party joins. Thus the meaning of a vote for any party depends upon what coalitions it is likely to enter, which in turn depends upon how other voters will vote" (Downs, 1957, p.163). Downs's model is described informally and not well-specified (in particular, there is no explicit equilibrium concept). His conclusions are correspondingly rather vague and difficult to verify (e.g. "1. Though rational voting is more important in multiparty systems than in two-party systems, it is more difficult and less effective. 2. In systems normally governed by coalitions, voters are under pressure to behave irrationally ... " (p.143)). Nevertheless, Downs does invoke an explicit two-stage model of legislative elections, and is sensitive to the additional strategic considerations this involves.

Given a set of parties C contesting the election, the election rule he states (p.144) is as follows. Take the aggregate electoral vote, n , and divide by K , the size of the legislature. Divide this number into each party's votes to obtain the number of legislative seats for that party. Since Downs ignores fractions, I shall refer to the weight (ω_k) of a party in the legislature rather than its number of seats, which may not be well-defined (see Balinski and Young, 1982). Hence, for any party k , $\omega_k(\sigma) = Kv_k(\sigma)/n$. Unfortunately, Downs's scheme amounts to admitting any party with at least one electoral vote into the legislature. This is trivial. However, by requiring that a party obtain sufficient electoral votes to get a weight of at least one suggests the following, nontrivial, election

rule:

$$\epsilon_C^d(\sigma) = \{y_k \in C \mid v_k(\sigma) \geq n/K, K \in \mathbb{N}\}.$$

Once the legislature C^* is determined, it "selects a prime minister by majority vote and approves his government department heads as a group before they start to govern. ... there are no intermediate votes between the initial approval of a government and the next election, either by the legislature or by the voters" (p.144). It is not clear from Downs's account whether he envisages a cooperative or a noncooperative majority voting game in the legislature. The description of the process indicates a noncooperative model but, immediately following this description, he writes that "we ignore most of the problems caused by interparty negotiations within the legislature, since they are both too complex and too empirical to be handled here" (p.145). This disclaimer suggests that Downs believes a cooperative model is more appropriate, but chooses to adopt a noncooperative voting framework in order to say anything at all. Certainly, the analysis following these statements makes most sense from the noncooperative perspective. So assuming this is indeed his approach, the legislative outcome is determined by weighted majority voting over C^* , with the weights given by $\omega(\sigma) = (\omega_k(\sigma))_{C^*}$. Thus,

$$\lambda_C^d(C^*, \sigma) = \{y \in C^* \mid y \text{ is the outcome of (noncooperative) weighted majority voting over } C^*, \text{ with weights } \omega(\sigma)\}.$$

The legislative election structure $(\epsilon_C^d, \lambda_C^d)$ is admissible, and is therefore not straightforward (Theorem 1). Hence Downs's concern throughout the chapter with individuals' voting decisions.

However, it is worth noting that if we take Downs's original specification of the (trivial) election rule -- i.e. $\epsilon_C^{d'}(\sigma) = \{y_k \in C \mid v_k(\sigma) > 0\}$ -- then the reduced form $\gamma_C^{d'}(\cdot) = \lambda_C^d(\epsilon_C^{d'}(\cdot), \cdot)$ is straightforward. In other words, given the noncooperative interpretation of Downs legislative model and given the (trivial) election rule $\epsilon_C^{d'}$, his concern with how rational individuals should vote is misplaced: their equilibrium voting strategies are sincere. Since I have argued that Downs,

for analytical reasons, favours a noncooperative legislative voting model to select the government, it is necessary that his election rule be nontrivial for there to be an issue over electoral voting behaviour in his model.

Downs does not resolve how voters should behave in any (voting) equilibrium. In the absence of a solution to this problem, parties' electoral strategies -- choice of platform -- cannot be pinned down, since the mapping taking a set of positions C into final party payoffs is then not well-defined. Following Downs's lead -- but failing to heed his warning that problems of "interparty negotiations within the legislature [are] both too complex and too empirical to be handled ..." -- Austen-Smith and Banks (1987) develop a multi-stage legislative election model under proportional representation, in which voters are fully rational and final policy outcomes emerge as equilibria to a fully specified legislative bargaining game.

In the models due to Greenberg et.al., above, the number of parties who can enter the legislative election is constrained only by the cardinality of X (no two parties are allowed to share the same platform). Austen-Smith/Banks permit parties to adopt any position in X (including those already occupied by some other party), but assume that only three parties contest the election -- hence, elections involve only $C \in C_3$. The election rule is $fs(m)$, $3 \leq m < n/3$, m and n odd.

The model is one of proportional representation and the weight of any party y_k in the post-election legislature is given by that party's vote-share, $v_k(\sigma)/n$.⁸ The importance of party weights lies in the specification of the legislative outcome function. Let C^* be the legislature (the set of parties with $v_k(\sigma) \geq m$) and assume that any coalition in C^* with aggregate vote-share exceeding $1/2$ is winning, i.e. has control of the legislative decision-making process.⁹ Call any winning coalition a government, and consider the following legislative bargaining game.

Suppose $|C^*| = 3$. If any party has a vote-share greater than $1/2$ then that party has monopolistic control of the legislature, and implements its electoral policy platform. If no party has such a majority, then the party with the highest vote-share is first given an opportunity to form a

government. It does this by proposing a winning coalition, a policy, and a distribution of portfolios (discussed below). If the number of parties in the proposed coalition who agree to the proposal is sufficient to form a government, then that government forms and implements the specified policy. If a government fails to form, then the party with the second-highest vote-share is given the opportunity to form a winning coalition. Should this fail, the smallest party in the legislature attempts to form a government. And, finally, if no party is able to support a winning coalition, a "caretaker" government forms which is presumed to make the legislative decisions "equitably". Now suppose $|C^*| \leq 2$. Then if only one legislative party has the highest weight, then that party has monopolistic control of the legislature (even if it does not have an overall electoral majority). Otherwise, the process above is implemented. As before, ties in vote-shares are broken by a fair random device allocating a decisive vote prior to the start of the bargaining process.

This type of mechanism is common among multi-party legislatures. For example, it occurs by convention in Britain, Belgium, Canada and Italy, and by law in Israel.¹⁰

The payoffs to parties generated by this process are discussed shortly. Once these are specified, the process of forming a government described here generates a noncooperative sequential bargaining game, in which parties' opportunity costs from joining a proposed government are generated endogenously. Parties' strategies in this (legislative) game are proposals, i.e. triples (D, y_D, b_D) , where D is a winning coalition, y_D is the policy position implemented by D , and b_D is the distribution of portfolios implemented by D .¹¹ The equilibrium concept used to solve the game is subgame perfect Nash equilibrium: call any such equilibrium a C^* -legislative equilibrium. Then the LOF is,

$$\lambda_C^+(C^*, \sigma) = \{y \in X \mid y \text{ is a component of some } C^*\text{-legislative equilibrium outcome}\}.$$

It turns out that $\lambda_C^+ \in \Lambda^*$ (given $|C| = 3$). Therefore, since $fs(m) \in \Sigma^*$ for $m \geq 3$, the legislative election structure is admissible relative to C_3 .

Although voters are purely policy-oriented (as described in section 2), parties care both about

policy and about portfolios. A legislative party's portfolio is modelled as that party's share of some transferable resource, B . Thus: $b_D = (b_{Dk})_{k \in C^*} \in B^{|C^*|}$, $b_{Dk} \geq 0$ all $k \in C^*$, $\sum_{C^*} b_{Dk} \leq B$. Fix a set of electoral positions for parties, $C = \{y_1, y_2, y_3\}$. Then party k 's payoff from any outcome of the legislative bargaining game involving a policy $y \in X$ and a portfolio distribution $b \in B^3$ is given by:

$$\Pi_k(C, B; \sigma) = \begin{cases} b_k - (y - y_k)^2 & \text{iff } k \in C^* \\ -c & \text{otherwise} \end{cases}$$

where $c > 0$ is a cost paid by the party if it fails to be elected, i.e. $v_k(\sigma) < m$. The assumption of quasi-linear preferences is a convenience, allowing, for any $C \in C_3$ and any $\sigma \in C^n$, explicit computation of the set of C^* -legislative equilibria. Before discussing this set, the key features of party payoffs need some justification.

The first important feature is that portfolio is valuable: this is unexceptional. That its value is independent of policy outcome and of other parties' portfolios may not be so reasonable. Parties, for instance, might be expected to prefer holding a portfolio in a government which implements a relatively more-preferred policy, than one implementing a less-preferred position. Similarly, a party holding office in a government in which one of its coalition partners, say, is considered "ideologically unsound" by that party, is likely to value the office differently than otherwise. With such considerations, the trade-off between policy outcome and portfolio will be more complex. However, that there is a trade-off at all is the important observation (moreover, separability is not crucial to all of the properties of C^* -equilibria). In many formal treatments of legislative bargaining, the process is modelled as a cooperative game without side-payments (e.g. McKelvey, Ordeshook and Winer, 1978; Schofield, 1985). The solution concepts for such games generate families of coalitions and utility payoffs, rather than identifying particular coalition and policy outcomes. By introducing an institutional mechanism -- the legislative bargaining game -- along with a transferable resource -- portfolio -- the trade-off between portfolio and policy can be exploited to generate unique

equilibrium predictions.

The remaining key feature of party "preferences" is that a party's payoff declines as a function of the distance between its (strategically chosen) electoral policy platform, and the final outcome implemented by the government. As it stands, this is ad hoc. The motivation comes from parties and voters interacting over several elections. When there are repeated elections, individual voting strategies can be conditioned on the past behaviour of parties. Consequently, voters can "punish" a currently elected party in subsequent elections to the extent that legislative outcomes fail to match electoral promises (platforms). However, since the game is not repeated, there is, strictly speaking, no mechanism in the Austen-Smith/Banks model by which these costs can be realized. Instead, there is an implicit assumption that the one-shot game of their model sufficiently approximates one play of a more complex repeated election game.

Given the above legislative bargaining game, if $|C^*| \leq 2$ or if $v_k(\cdot) \geq (n+1)/2$ then the C^* -equilibrium outcome is trivially given by the largest party's policy position, and that party takes all the portfolio.¹² In the more interesting case, the following is true:

Theorem 6 [Austen-Smith & Banks, 1987]: For any $C \in C_3$ and any $\sigma \in C^n$ such that $m \leq v_k(\sigma) < (n+1)/2 \forall k$, there is a unique C^* -legislative equilibrium outcome (D^*, y_{D^*}, b_{D^*}) . It is such that:

- (a) $D^* = \{k, j \in C \mid v_k(\sigma) = \max_{C^*} \{v_h(\sigma)\}, v_j(\cdot) = \min_{C^*} \{v_h(\sigma)\}\}$,
- (b) $y_{D^*} \in [y_k, (y_k + y_j)/2]$; where k and j are defined in (a),
- (c) $b_{D^*h} = 0$ if $h \notin D^*$; $b_{D^*k} > 0$; $b_{D^*j} \geq 0$; where k and j are defined in (a).

With the exception of the bound $(y_k + y_j)/2$ in (b), Theorem 6 does not depend on the quadratic specification of Π_k . The result says that the winning coalition that forms the government is minimal winning (the coalition-of-the-whole does not form) but not minimum winning in the sense of Riker

(1962). The second largest party gets excluded from the coalition because of its bargaining power relative to the smallest party: when the largest party seeks a coalition partner, it can extract relatively more surplus from the weakest member of the legislature. The result also implies that the government is not necessarily connected in the sense of Axelrod (1970): there is no reason to presume the positions of the largest and the smallest parties are adjacent in X .

In (b) and (c) of Theorem 6, the exact location of the final outcome, and the particular equilibrium distribution of portfolios, depend on B and on the entire list of candidate positions, C . With this information and with specified payoff schedules, (D^*, y_{D^*}, b_{D^*}) can be computed precisely as a function of voting strategies, σ . Since voters are rational, voting strategies in turn are a function of the policy platforms, C . Hence, given B , C and Π_k , Theorem 6 shows that the LOF $\lambda_C^+(\cdot, \sigma)$ is a well-defined function of σ . This is crucial in specifying the behaviour of voters and parties at the electoral stage of the process.

Definition: A voting equilibrium for the legislative election structure $(fs(m), \lambda_C^+)$ is an n -tuple

$\sigma^*(u, C) \in C^n$ such that $\forall C \in C_3, \forall i \in N, \forall \sigma_i$:

$$Ew_i(\sigma^*(u, C)) \geq Ew_i(\sigma_i, \sigma_{-i}^*(u, C)),$$

where the expectation is over $\lambda_C^+(\cdot, \cdot)$.

Thus a voting equilibrium here is simply a Nash equilibrium in which individuals vote on the basis of final legislative outcomes, and not on candidate positions per se.

As observed in the Introduction, with two-party, simple plurality elections, sincere voting at the electoral stage is the only sensible Nash voting equilibrium for any pair of policy positions: although there exist many Nash equilibria in voting strategies, σ , the sincere strategy vector, σ^T , is the unique undominated equilibrium. This is true largely because, under simple plurality voting, a party's legislative influence is monotonic in its electoral vote-share. Unfortunately, by part (a) of Theorem

6, this is not true with multi-party elections with proportional representation. And it turns out in this environment that no individual voting strategy, given $C = \{y_1, y_2, y_3\}$, is dominated. So using dominance arguments to refine the set of voting equilibria in legislative election games achieves nothing. However, since the strategic choice of policy platforms with which to contest the election will depend, inter alia, on how voters respond, the issue of voting equilibrium selection is important.

The refinement used in Austen-Smith/Banks is to require that every voter in an electoral equilibrium is pivotal. Let $\sigma^{**}(u, C)$ be the selected voting equilibrium for C . Then:

Definition: An electoral equilibrium relative to $\sigma^{**}(u, \cdot)$ for the legislative election structure $(fs(m), \lambda_C^+)$, is any $C^0 \in C_3$ such that for every party k and every $y_k \in X$:

$$E\pi_k(\sigma^{**}(u, C^0)) \geq E\pi_k(\sigma^{**}(u, y_k, C^0 \setminus \{y_k^0\})),$$

where $\pi_k(\sigma^*(\cdot, \cdot)) \equiv \Pi_k(\cdot, \cdot, \sigma^*(u, C))$ is k 's final payoff from the legislative bargaining process, and the expectation is over $\lambda_C^+(\cdot, \cdot)$.

An electoral equilibrium is a Nash equilibrium in policy positions when parties rationally take account of voter behaviour and the subsequent legislative bargaining game. Prima facie, the refinement on voting equilibria offered above seems circular: in any electoral equilibrium, all voters are required to be pivotal, but the definition of electoral equilibrium is itself predicated on how voters will behave. However, there is no more difficulty here than that involved in solving a pair of simultaneous equations.

Given that voting strategies invariably constitute Nash responses, the refinement puts no restrictions on out-of-(electoral) equilibrium voting behaviour. But it does isolate a relatively small class of party positions as electoral equilibria. And given this, the selection of any out-of-electoral equilibrium voting strategies is more-or-less determined by the natural requirement that such

strategies provide incentives for parties to "move toward" an electoral equilibrium. (It is perhaps worth noting that sincere voting everywhere is not capable of supporting any electoral equilibrium.) Nevertheless, the general problem of voting equilibrium selection in multi-party legislative elections is nontrivial, and by no means solved. It is in this context that some form of cooperative model of voting -- such as Sugden's (1984) -- may prove valuable.

Assume that the distribution of voter preferences on X is symmetric, and that all voters have symmetric, strictly concave, utilities. For any $C = \{y_1, y_2, y_3\} \in C_3$, label parties so that $y_1 \leq y_2 \leq y_3$. Then the set of electoral equilibria identified is:

Theorem 7 [Austen-Smith & Banks, 1987]: $C = \{y_1, y_2, y_3\} \in C_3$ is an electoral equilibrium relative to $\sigma^{**}(u, \cdot)$ for the legislative election structure $(fs(m), \lambda_C^+)$, iff:

- (a) $y_2 = \text{median}\{x_1, \dots, x_n\}$,
- (b) $(y_2 - y_1) = (y_3 - y_2) \in [z, z']$; $z, z' \in \mathbf{R}$, $z' > z > 0$.

Furthermore, there is a unique Pareto efficient equilibrium (relative to $\sigma^{**}(u, \cdot)$) at $|y_k - y_2| = z$, $k = 1, 3$.

Electoral equilibria involve one party adopting the median voter's ideal point, and the remaining two parties locating symmetrically about this point. The bounds z and z' are not arbitrary. To illustrate this result and to see where the bounds come from, consider the following example.

Example 2: Suppose there are 101 voters, each with quadratic preferences on $X = [-200, 200]$. Let the ideal points be $x = (-50, \dots, -1, 0, 1, \dots, 50)$. Let the election rule be $fs(25)$, and assume party payoffs are given by the quadratics, Π_k . Then $z = 32$, $z' = 100$, and any $C = \{y_1, y_2, y_3\}$ such that $y_2 = 0$ and $-y_1 = y_3 \in [32, 100]$ is an electoral equilibrium. The possible final outcomes generated by some C^* -equilibrium to the legislative bargaining game when C is such a set of

platforms, are:

$$\cup_{\sigma} [\lambda_C^+(\cdot, \sigma)] = \{y_1, y_2, y_3, y_{12}, y_{23}\},$$

where $y_{hk} = [y_h + y_k]/2$. In the $\sigma^{**}(\cdot, C)$ voting equilibrium, individuals $i = 1, \dots, 38$ vote for y_1 ; individuals $i = 39, \dots, 63$ vote for y_2 ; and individuals $i = 64, \dots, 101$ vote for y_3 . Hence, $v_1(\sigma^{**}) = v_3(\sigma^{**}) = 38 > v_2(\sigma^{**}) = 25$, and all voters are pivotal. Notice that in any electoral equilibrium for this environment, at least individuals $i = 36, 37, 38, 64, 65, 66$ do not use a sincere voting strategy at σ^{**} : as y_1 and y_3 move outwards toward (respectively) -100 and 100, the number of individuals voting strategically increases. By Theorem 6, the final policy outcome is $y_{D^*}(\sigma^{**}) \in \{y_{12}, y_{13}\}$, with each outcome occurring with probability one-half.

If $|y_k - y_2| \geq z' = 100$, then the number of individuals who strictly prefer y_2 to any other alternative in $\cup_{\sigma} [\lambda_C^+(\cdot, \sigma)]$ exceeds 50. In this case, an overall majority of the electorate prefers that party 2 has monopolistic control of the legislature -- there is no C^* -equilibrium outcome which is preferred by them. Consequently, the voting equilibrium selection made for this case is that everyone votes sincerely relative to C . This makes good sense, and prevents parties from being "too dispersed" in electoral equilibrium, relative to the distribution of voter preferences. And notice that the unique efficient electoral equilibrium is when the parties are minimally dispersed (within the set of equilibria): this is immediate from inspection of Π_k . The inner bound z is generated implicitly by the requirement that at least $m = 25$ individuals must find it a best-response to vote for y_2 , given that others' votes imply $v_1 = v_3$. In other words, these individuals must prefer the fair lottery over $\{y_{12}, y_{32}\}$ to a certain outcome of y_1 or y_3 . Evidently, the closer a voter is to y_1 or y_3 , the greater is his incentive (given risk-aversion) to vote for the relatively extreme party when he is pivotal. So the bound z is computed by finding the individual who is just indifferent between the lottery and a certain outcome of the platform of the extreme party closest to him or her.

There are three substantive features of electoral equilibria illustrated in Example 2 which are not

special to the example. First, the middle party receives the smallest vote-share and the others receive the same share; second, not everyone votes sincerely, so that the distribution of vote-shares in the legislature does not reflect the distribution of voter preferences; and third, realized final policy outcomes are skewed away from the median and so do not reflect the relative weights of parties in the legislature. Virtually all advocates of proportional representation base their arguments on the premises that legislative representation will better reflect the diversity of electorate preferences, and that final policy outcomes will correspondingly reflect relative party weights in the legislature. The results of Austen-Smith and Banks suggest that this base is rather fragile.

To my knowledge, the papers above exhaust the set of formal models explicitly concerned with single-district legislative elections (as defined in the Introduction). Related work is due to Gary Cox (1984a, b), who studies an electoral model of multi-candidate, double-member districts. Cox's concern is not with the election of an entire assembly, but with how equilibrium candidate platforms under a double-member district system might differ from those under the canonic, single-member district, system. From this perspective, it is natural to maintain the implicit assumption of the canonic model; viz. that agents in a single district election behave as if there were only one district. By strengthening this assumption (as Cox does not) to presuming that there is indeed only one district in the polity, then the fact that more than one candidate is to be elected from the district makes Cox's electoral model "legislative" in the sense of this paper. However, voters' strategy sets are somewhat different to those considered up to now: with double-member districts, voters are permitted to vote for at most two candidates.¹³

For any $C \in \mathcal{C}$, let $\underline{\sigma}_i: U \times C \rightarrow (C \cup \emptyset)^2$ describe individual i 's voting strategy, where, for example, $\underline{\sigma}_i(u_i, C) = \{y_k, \emptyset\}$ means i votes only for candidate y_k , and $\underline{\sigma}_i(u_i, C) = \{y_k, y_j\}$ means i votes for candidates y_k and y_j . Write $\underline{\sigma}(u, C) = (\underline{\sigma}_1, \underline{\sigma}_2, \dots, \underline{\sigma}_n)$. Individuals are assumed have Euclidean preferences, so the analogue of sincere voting for any individual i in this environment is that i casts two votes, one for each of the two candidates closest to that voter's ideal point in X , an

interval (ties are broken by flipping a fair die). Let $\underline{\sigma}^T(u, C)$ be the strategy vector in which every individual uses this sincere strategy ($u \in U_e^n$).

In the 1984a paper Cox assumes $C \in \mathcal{C}_t$, $t = 3, 4$. For any candidate $k \in C$, define,

$$M_k(\underline{\sigma}^T(\cdot, C)) = \begin{cases} v_k(\underline{\sigma}^T(\cdot, C)) - \min.\{v_j(\underline{\sigma}^T(\cdot, C))\}_{j \neq k}, & \text{iff } C \in \mathcal{C}_3 \\ v_k(\underline{\sigma}^T(\cdot, C)) - \text{median}.\{v_j(\underline{\sigma}^T(\cdot, C))\}_{j \neq k}, & \text{iff } C \in \mathcal{C}_4 \end{cases}$$

to be k 's margin under $\underline{\sigma}^T(\cdot, C)$, given candidate platforms, C . "For the two winning candidates, this $[M_k(\cdot)]$ is their margin of victory. For the losing candidate[s], it indicates how many votes short of a seat he or she is [they are]" (Cox, 1984a, p.446). The election rule is $\text{fn}(2)$, where it is understood that in the definition of $\varepsilon_C^{\text{fn}(K)}$ the strategy vector σ is replaced by $\underline{\sigma}$.

Definition: C^0 is an fn(2) Nash equilibrium for the double-member election game on \mathcal{C}_t , $t = 3, 4$, iff $C^0 \in \mathcal{C}_t$ and, for every party k and every $y_k \in X$, $M_k(\underline{\sigma}^T(u, C^0)) \geq M_k(\underline{\sigma}^T(u, y_k, C^0 \setminus \{y_k^0\}))$.

Given $u \in U_e^n$, let $z_1 = \sup.\{z \in X \mid 2 \cdot (\#\{i \in N \mid x_i \leq z_1\}) < \#\{i \in N \mid x_i > z_1\}\}$, and let $z_2 = \inf.\{z \in X \mid 2 \cdot (\#\{i \in N \mid x_i \geq z_2\}) < \#\{i \in N \mid x_i < z_2\}\}$. Loosely speaking, z_1 and z_2 are the quantiles of order 1/3 and 2/3, respectively.

Theorem 8 [Cox, 1984a]: (a) $C^0 \in \mathcal{C}_3$ is an fn(2) Nash equilibrium iff $y_k^0 = y^* \in [z_1, z_2]$, $k = 1, 2, 3$. (b) Suppose n is odd. Then, $C^0 \in \mathcal{C}_4$ is an fn(2) Nash equilibrium iff $y_k^0 = \text{median}.\{x_1, \dots, x_n\}$, $k = 1, 2, 3$.

So in both cases (3 and 4 candidates), the classical Downsian convergence result reemerges. But, unlike the 4 candidate case in which the median voter defines the only equilibrium, with only 3 candidates many equilibria in addition to the median voter position are feasible. This last is a nice

result, since in three candidate competition for a single-member district with sincere voting, no Nash equilibrium exists (see, e.g., Cox, 1987). But as a model of legislative elections, Theorem 8 suffers from the two familiar difficulties: no legislative outcome function is defined (so that voters vote over candidates, not policy outcomes); and voters are constrained to vote sincerely. In any fn(2) Nash equilibrium parties converge, so one might argue that sincere voting is justified. But this would be wrong since the equilibria are supported by sincere voting at out-of-equilibrium party platforms. If we relax the requirement of sincerity, it is not evident that Theorem 8 obtains.

Cox is clearly aware of the problem of strategic voting. In his 1984b paper, he fixes a set of candidates C arbitrarily, and examines an arbitrary individual's (noncooperative) strategic voting calculus. By taking C as given, the legislative outcome function can be left implicit: individuals' preferences over candidates can be assumed to be induced from candidate positions and some underlying LOF. Cox explicitly adopts this interpretation of preferences. For the case of $C \in C_3$ and voting strategies for each individual described by $\sigma_i(\cdot, C)$, the situation is "essentially equivalent to approval voting with three candidates; thus, voting in double elections with three candidates is "sincere" in the same sense that approval voting is ... Nonetheless, it is shown that electoral choice in double (and approval) elections is inherently strategic in the sense that voters' beliefs about how others will vote affect their decisions ... " (Cox, 1984b, p.737).¹⁴

Having said this, however, it is worth pointing out that an individual's beliefs in Cox's analysis are arbitrarily assigned subjective estimates of others' behaviour. Consequently, there is no guarantee that these beliefs are "rational" in the sense that, when all individuals play the strategies prescribed by their beliefs, everyone's beliefs are necessarily consistent with the observed electoral outcome. In other words, it is an open question within Cox's framework whether his conclusions hold in an appropriately defined Bayesian voting equilibrium. Nevertheless, earlier arguments (Theorem 1 ff.) suggest they will hold. And Cox finds some empirical evidence at an aggregate level to support the claim of strategic voting in double elections.

5. Multi-district models

When the electorate is partitioned into several districts, with each district periodically electing representatives to a national legislature, it becomes necessary to distinguish "candidates" from "parties". A party comprises at least one candidate for office, and if different candidates of a given party contest distinct districts, then there is no reason a priori to assume that they would wish to fight an election on precisely the same platform. This in turn raises the question of what it is to be a party: a question I shall, to all intents and purposes, sidestep for the present.

Throughout this section, assume the electorate N is partitioned into a finite number, $1 < K \leq n$, of districts: $\mathcal{P}(N) = \{ \{N_r\}_{r=1, \dots, K} \mid N_r \neq \emptyset; (N_{r'} \cap N_r) = \emptyset, r' \neq r; (\cup_r N_r) = N \}$; let $|N_r| = n_r$. Also assume that, in any legislative election, exactly two candidates contest each district under a single-member district, simple plurality election rule.¹⁵ To define this formally, let $C_r = \{y_{ar}, y_{br}\} \in C_2$ be the pair of candidate platforms offered in district $r \in \Delta = \{1, \dots, K\}$. (As we shall see later, these may or may not be party platforms.) For any individual $i \in N_r$, i 's voting strategy is a map, $\sigma_i^r : U \times C_r \rightarrow C_r$. Let $\sigma^r(u_r, C_r) = (\sigma_1^r(\cdot, C_r), \dots, \sigma_{n_r}^r(\cdot, C_r))$, where u_r is the restriction of u to N_r . Let $C = \cup_r C_r \in C_{2K}$, and write $\sigma(u, C) = (\sigma^r(u_r, C_r))_{r \in \Delta}$. Then the election rule assumed throughout section 5 is:

$$\varepsilon_C^K(\sigma) = \{(y_{pr})_{p \in \{a,b\}}, r \in \Delta \in \prod_r C_r \mid \forall pr, v_{pr}(\sigma^r) > v_{p'r'}(\sigma^r), p, p' \in \{a,b\}, p \neq p'\}.^{16}$$

While this rule is prevalent among Anglo-American political systems, it is not the only election scheme used in multi-district polities; and the assumption of only two candidates per district is restrictive. However, to my knowledge, there are no formal multi-district analyses of rules other than simple plurality.¹⁷ And results for multi-candidate elections with simple plurality voting, even in the single-district context, are delicate and hard to come by (Cox, 1987). Consequently, since one rationale for studying models of legislative elections is to explore the robustness of conclusions derived from the canonic (two-candidate, single-district, simple plurality) model, the focus on

$\epsilon_C^K(\cdot)$ is a natural starting point.

In chapter 2 of *A Theory of Party Competition*, David Robertson (1976) sketches a multi-district model in which candidates all belong to various nationally-oriented parties, and legislative outcomes are determined by the party whose candidates win a majority of districts. Implicit in his discussion is that all electorally successful members of a party can be relied upon to vote as a bloc in the legislature, and that there are only two parties. If either of these assumptions were not met, Robertson's conclusion that "The official party policy [i.e. that of the national organizers] will be the one calculated to win a majority of constituencies, not the one which has majority support in the electorate at large" (p.52), would be untenable. He is, however, explicit in arguing that rational voters in such polities will vote on the basis of party policy and not on the basis of candidate policy, whenever these are distinct. From the theoretical perspective this is clearly correct since, given the implicit assumptions just mentioned, it amounts to saying that voters vote over final policy outcomes and not over candidates' positions per se.

His model is not rigorous and his results are "broad brush" (e.g. "No great degree of autonomy will be enjoyed by candidates in presenting policy" (p.53)). Taking Robertson's cue, Austen-Smith (1984, 1986, 1987a) examines a multi-district model of electoral competition under $\epsilon_C^K(\cdot)$ in which there exist exactly two parties or none (i.e. all candidates for legislative office run as independents).¹⁸ A party is any set of candidates, each member of which shares a common label (e.g. Republican, Democrat, Conservative, Labour). If there are parties, these are labelled $p = a, b$, and it is assumed that each party consists of K candidates: thus, y_{pr} denotes the electoral platform adopted by the candidate of party p contesting district r . When there are no parties, the index p simply distinguishes the contestants for any district. For the moment, assume all candidates for office belong to one or other of the two political parties, p .

Under $\epsilon_C^K(\cdot)$ the legislature consists of K elected officials, one from each district. Assume (for expository convenience) that K is odd. A winning coalition in an elected legislature is any subset D

of $\epsilon_C^K(\cdot)$ such that $|D| \geq (K+1)/2$. Let $D(C, \sigma) = \{D \mid \epsilon_C^K(\sigma) \supseteq D, |D| \geq (K+1)/2\}$ be the set of winning coalitions (governments) in the legislature given the set of candidates, C , and voter strategies, σ . Which particular government will form and what policy outcome it will implement depends on the details of legislative bargaining etc.. Unlike the single-district, multi-party environment, the assumption of many districts and two political parties suggests a natural approach to finessing problems of legislative coalition formation and bargaining.

Definition: A constitution for party p is a map, $g_p : C_K \rightarrow X$, such that $\forall C \in C_K, g_p(C) \in co.(C)$.

A constitution for $p \in \{a, b\}$ defines how the list of the party's candidates' policy platforms are mapped into a legislative party policy, which is constrained to lie in the Pareto set of the party members' positions. Examples of party constitutions (PCs) are voting within the whole party (say, at conference), intra-party bargaining outcomes, dictatorship of the party leader, and so on and so forth. As defined, however, the party policy given by g_p is a function of all of the party's candidates' electoral positions. So, for instance, the use of some within-party voting rule for which the franchise is extended only to the legislative (i.e. winning) party members is excluded. This is an important substantive restriction. It amounts to assuming that once all party members have announced their platforms, $C^p \in C_K$, the party is committed to implementing the policy $g_p(C^p)$ should it gain control of the legislature. Having said this, and although PCs are treated as exogenous here, various assumptions about party structure can be introduced through restrictions on the maps g_p .

For each party p , let $p(C, \sigma)$ denote the set of party p candidates who get elected to the legislature; i.e. $p(C, \sigma) = \{y_{pr} \in C \mid v_{pr}(\sigma^r) > v_{p'r'}(\sigma^r), p \neq p'\}$. Consider the following legislative

outcome function:

$$\lambda_C^P(\cdot, \sigma) = \{y \in X \mid y = g_p(y_{p1}, \dots, y_{pK}), p(C, \sigma) \in D(C, \sigma)\}.$$

Since K is odd and all candidates belong to one of two parties, $\lambda_C^P(\cdot)$ is well-defined.

Given party constitutions, the legislative election structure $(\epsilon_C^K, \lambda_C^P)$ models two-party, multi-district, first-past-the-post electoral systems in which legislative members of any party fully coordinate their legislative behaviour to promote a given (through the PC) party policy position. (There is no presumption that at the electoral stage candidates similarly coordinate their strategic choices.) Empirically, this corresponds best to British-type parliamentary systems.

Although the LOF λ_C^P is anonymous with respect to parties, it is not anonymous with respect to candidates: party labels matter. Theorem 1, therefore, does not apply. Nevertheless, $(\epsilon_C^K, \lambda_C^P)$ is not straightforward. This is evident: once party constitutions are defined, rational voters will (as Robertson observes) vote over party policies, and not candidate positions per se.

Consequently, both of the following can hold for $i \in N_i$:

$$u_i(g_a(y_{a1}, \dots, y_{aK})) > u_i(g_b(y_{b1}, \dots, y_{bK})) \text{ and } u_i(y_{ar}) < u_i(y_{br}).$$

It is the first inequality which is relevant for a rational voting decision, not the second. Indeed, given the two parties' candidates electoral platforms, $C^a = \{y_{ar}\}_r$ and $C^b = \{y_{br}\}_r$, and given the legislative election scheme $(\epsilon_C^K, \lambda_C^P)$, all voters have a weakly dominant strategy: vote sincerely over party policies $\{g_a(\cdot), g_b(\cdot)\}$. This voting rule is left implicit hereafter.

Given candidates' electoral platforms, $C = (C^a, C^b)$, let the probability that party p wins r districts, given voter strategies $\sigma(C)$, be $q_r^p(\sigma(C^a, C^b)) \in [0, 1]$.

Definition: Party p is Downsian iff (1) $y_{pr} = y_p, \forall r$; and (2) $y_p = \arg\max_X \sum_r r \cdot q_r^p(\sigma(y, C^p))$.

Downs (1957) assumes a single district polity, and makes an assumption on party structure which "In effect ... treats each party as though it were a single person" (p.26). He then supposes that parties strategically select their electoral policy platforms to win control of the legislature. With no abstention, rational voting and simple plurality elections, winning control of the legislature amounts to maximizing votes. All of this is defensible in the single district framework. But in a multi-district

model, parties necessarily consist of several candidates, and maximizing votes is not a sensible strategy for winning legislative control. The definition of a Downsian party above is the analogue, for the multi-district two party polity under $(\epsilon_C^K, \lambda_C^P)$, to Downs's original conception. Part (1) of the definition says that the party is fully coordinated in its electoral strategy, given that voters are rational and vote on final outcomes under the legislative outcome function, λ_C^P . Part (2) of the definition says that the party's electoral platform is selected to win control of the legislature. Note that the PC for any Downsian party is trivial: since $y_{pr} = y_p$ for all r , $g_p(\cdot) \in \text{co.}(C^p) \Rightarrow g_p(\cdot) = y_p$.

Definition: A Downsian equilibrium under $(\epsilon_C^K, \lambda_C^P)$ is a pair $(y_a^*, y_b^*) \in X^2$ such that,

$$(1) \forall r, y_{ar}^* = y_a^* \text{ and } y_{br}^* = y_b^*;$$

$$(2) \forall y \neq y_p^*; \forall p = a, b; \sum_r r \cdot q_r^p(\sigma(y_p^*, y_{p'}^*)) \geq \sum_r r \cdot q_r^p(\sigma(y, y_{p'}^*)); p' = a, b; p \neq p'.$$

In the definition of a Downsian party, there is an alternative objective a party seeking legislative control under $(\epsilon_C^K, \lambda_C^P)$ might adopt. Instead of assuming expected district maximization, we could suppose a party maximizes the probability of winning a simple majority of districts; i.e. replace (2) in the definition of a Downsian party by,

$$(2') y_p = \arg\max_X q_{(K+1)/2}^p(\sigma(y_p, C^p)), p \neq p'.$$

However, given two parties and rational voters, it is not hard to check that in the one-dimensional environment assumed throughout, (y_a^*, y_b^*) is a Downsian equilibrium if and only if it is also an equilibrium in which (otherwise Downsian) parties choose y_p according to (2') rather than (2). So we lose no generality on this count by assuming, for Downsian parties, (2) rather than (2') hereafter.

Theorem 9 [Austen-Smith, 1987a; Hinich & Ordeshook, 1974]¹⁹: $(y_a^*, y_b^*) \in X^2$ is a Downsian equilibrium under $(\epsilon_C^K, \lambda_C^P)$ iff $y_a^* = y_b^* = \mu^*$; where $\mu^* = \text{median}\{\mu_T\}_T$, and μ_T is

the median voter's ideal point in district N_r .

Theorem 9 is the natural generalization of Downs's classical result: Downsian parties in a multi-district polity with rational, final policy oriented voters, converge on the median of the medians. Now Robertson's basic model (1976) is essentially one of two party competition under $(\epsilon_C^K, \lambda_C^P)$ in which both parties are Downsian.²⁰ This result, then, casts light on some of his claims. First observe that, because parties converge in equilibrium and because voters are purely final (i.e. party) policy oriented, each party has a 50/50 chance of winning control of the legislature. Similarly, every candidate also has a 50/50 chance of winning his or her electorate. Hence, Robertson is not correct in claiming that "to allow that constituencies may not all have the same distribution of ideological opinions [i.e. $\mu_r \neq \mu_{r'}$, all r, r'] is to entail that if party [p] is to fight an election on a constituency maximizing platform, some of its candidates are doomed to fight individual seats on platforms that cannot win" (p.50). For this conclusion to be correct in equilibrium, voters cannot be purely final policy oriented. On the other hand, Robertson is right to argue that there will be tension between a party (i.e. national organizers) and at least some of its office oriented candidates. Label districts so that $\mu_r < \mu_{r+1}$, all $r < K$. Then, say, given $y_a^* = \mu^*$, candidates of party b contesting districts $r > (K+1)/2$ have an incentive to move y_b^* to the right because this guarantees that they will win legislative office under the equilibrium voting strategy (sincere voting over party platforms). The assumption of a party being Downsian, however, implies that such candidates cannot do this (Robertson's view) or are not office oriented. Finally, since there is no reason to suppose $\text{median}\{x_1, \dots, x_n\} = \mu^*$, there is no reason to suppose that "the winning electoral strategy [in the Downsian equilibrium] is that of the national median voter's preference" (ibid.).

Political parties do not have objectives. It is individuals who are purposive, and parties are collections of individuals. From this perspective, the (implicit) Downsian party constitution is

extremely restrictive. Wider classes of party organization can be accommodated within $(\epsilon_C^K, \lambda_C^P)$ legislative structures, and one of these is studied by Austen-Smith (1984).

Definition: A party p is **Type I** iff (1) $\forall pr$, g_p is continuous in $y_{pr} \in X$ and differentiable in $y_{pr} \in \text{interior}(X)$; and (2) every party candidate pr seeks to win his or her own district -- in the absence of abstention, this amounts to within-district vote-maximization by pr.

So unlike in Downsian parties, candidates of Type I parties do not coordinate their electoral policy choices (save implicitly through the PC) to promote any party objectives per se.

For any subset of districts L in Δ , and any set of party p strategies $C^p \in C_K$, let $C^{pL} = \{y_{ps} \mid s \neq L\}$.

Definition: A **Type I party equilibrium** under $(\epsilon_C^K, \lambda_C^P)$ is a $2K$ -tuple $(C^{a*}, C^{b*}) \in C_{2K}$, $C^{p*} = \{y_{pr}^*\}_r$, such that for each $p \in \{a, b\}$, $\forall \{L \mid \Delta \supseteq L\}$, $\forall r \in L$, $\forall \{y_{ps}\}_L \in C_{|L|}$,

$$v_{pr}(\sigma(g_p(\{y_{ps}^*\}_L, C^{pL*}), g_p(C^{p*}))) \geq v_{pr}(\sigma(g_p(\{y_{ps}\}_L, C^{pL*}), g_p(C^{p'*}))), p \neq p'.$$

A Type I party equilibrium is therefore a list of strategies, one for each candidate of each party, such that, given the policy platforms of all other candidates for office, no group of candidates within any particular party can coordinate their electoral policy platforms to improve their respective within-district vote totals. If $|L| = 1$, then this is simply a Nash equilibrium in which each candidate plays against the other party and against other members of his own party, via the PC. If $|L| > 1$, then the equilibrium concept is a form of Strong Nash equilibrium. On the one hand, it is somewhat stronger than Strong Nash since it requires that nobody benefits from any coalitional deviation. On the other hand, the only coalitions that are permitted to form are those involving candidates from within the same party. This last restriction seems natural in the present setting, in which candidates are

exogenously endowed with a party affiliation and the party constitutions are prespecified. In the absence of these assumptions, the restriction is less reasonable. Finally note that although candidates of a Type I party are not constrained to offer identical electoral platforms, under the legislative outcome function λ_C^P (and given the definition of a party constitution), successful Type I party members (i.e. those that get elected) still act as a bloc in supporting the party policy, $g_p(\cdot)$, in the legislature. Thus the voter strategies described earlier remain equilibrium (voting) strategies.

Suppose hereafter that voters' preferences are Euclidean, $u \in U_e^n$. The following result makes clear the tension between candidates' objectives and party structure mentioned by Robertson.

Theorem 10 [Austen-Smith, 1984]: $(C^{a*}, C^{b*}) \in C_{2K}$ is a Type I party equilibrium under $(\epsilon_C^K, \lambda_C^P)$ iff,

- (a) $g_a(C^{a*}) = g_b(C^{b*}) = y^*$;
- (b) $\forall p, \forall L$ in Δ s.t. $[r \in L \Rightarrow \mu_r < (>) y^*]$, $\forall \{y_{pr}\}_L$: $g_p(C^{p*}) \leq (\geq) g_p(\{y_{pr}\}_L, C^{pL*})$.

Theorem 10 claims that, as in Downsian party competition under $(\epsilon_C^K, \lambda_C^P)$, party policies will converge in any Type I equilibrium, but in general the same is not true of candidate positions. In particular, candidates contesting districts with medians to the left of the common party position, y^* , will seek to move the party platform leftwards; and similarly for candidates competing for districts with medians to the right of y^* . Thus any Type I equilibrium will be associated with a dispersion of candidate positions within each party. And since there is no reason to expect the two parties' PCs to be the same, there is likewise no reason to suppose opposing candidates within districts converge -- to the median or anywhere else. But despite this lack of convergence at the local level, each candidate in equilibrium continues to enjoy a 50/50 chance of winning office. Again, this is because rational voters vote over party policies and these do converge.

Unfortunately, Theorem 10 is not an existence result for Type I party competition under $(\epsilon_C^K, \lambda_C^P)$,

λ_C^P). To insure that there is a suitable set of candidate positions satisfying both parts of the result, additional restrictions on party constitutions are required. For any set of candidate positions CP , let $y(CP)$ be the average platform within the party; $y(CP) = (\sum_r y_{pr})/K$.

Definition: A party constitution g_p , (1) discounts extremists iff $\exists t_p(\emptyset(N)) \in \mathbf{R}_{++}$ such that $\forall r \in \Delta$, $\forall CP = \{y_{pr}\} \cup CP \setminus \{r\} \in C_K$,

$$|y_{pr} - y(CP)| > t_p(\cdot) \Rightarrow [\exists y \in X \mid |y - y(CP)| \leq t_p(\cdot) \ \& \ g_p(CP) = g_p(y, CP \setminus \{r\})].$$

(2) is strongly symmetric iff g_p is symmetric with respect to candidates, and

$$[CP \text{ symmetrically distributed on } X] \Rightarrow [g_p(CP) = y(CP)].$$

The first condition says that although a candidate is free to adopt as extreme a position as he or she wishes relative to other party members' positions, the PC will give no more weight to this position in arriving at the party platform than it would give to a less extreme position. And exactly what constitutes "extreme" depends on the partition of the electorate (in particular, the distribution of medians, $\{\mu_r\}$). Assuming a party constitution discounts extremists seems reasonable (although ideally it is something we would like to explain, not just assume). Assuming a party constitution is strongly symmetric is not so reasonable: it rules out, for instance, the presence of disproportionately influential members in the party.

Theorem 11 [Austen-Smith, 1984]: Suppose g_p discounts extremists and is strongly symmetric, $p = a, b$. Then there exists a Type I party equilibrium under $(\epsilon_C^K, \lambda_C^P)$. Moreover, the equilibrium is unique in party policies with $g_p(C^{p*}) = \mu^*$, $p = a, b$; but the equilibrium is not necessarily unique in candidate platforms.

In any of the Type I equilibria under $(\epsilon_C^K, \lambda_C^P)$ identified by Theorem 11, party platforms

converge on μ^* , the median of medians $\{\mu_T\}$. Thus the final legislative outcome here exactly corresponds to that supported by Downsian party competition (Theorem 9): given rational policy oriented voters, the existence of self-seeking and (electorally) autonomous candidates is not necessarily incompatible with parties maximizing expected districts won.

However, this result does depend on assuming PCs discount extremists and are strongly symmetric. And although these conditions are not logically necessary for existence, they are minimally sufficient in the sense that, if either one is relaxed in the presence of the other, examples of polities for which no Type I equilibrium exists become easy to construct (Austen-Smith, 1984, pp.195-8). Essentially, the conditions insure a balance across candidates competing for districts with medians on opposite sides of μ^* . So one can interpret the Theorem as saying that within-party power must be evenly distributed to obtain stable electoral outcomes; in other words, disproportionately influential party members can be disruptive. Given this, Remark (3) of Greenberg and Shepsle (1987) must be treated cautiously. When discussing the implications of their result (Theorem 4, above), they write that "Since for every given society a [fn(1)-equilibrium] exists ... a society choosing K legislators by first partitioning itself into K constituencies ... and then empowering each constituency to choose a single representative produces an equilibrium ... , whereas that same society choosing K legislators "at large" would not in general produce an equilibrium" (p.). The first part of this claim rests crucially on their assumption of sincere voting and the lack of any specification of a legislative outcome function. As Theorem 11 shows, even when the candidates are tied to one of two parties and these are such that there exist only two possible final outcomes $\{g_a(\cdot), g_b(\cdot)\}$, existence of an electoral equilibrium is generally not guaranteed by the method they suggest.

An example of a party constitution satisfying the conditions of Theorem 11, and for which the Greenberg/Shepsle claim is correct, is: $g_p(C^P) = \text{median}\{y_{p1}, \dots, y_{pK}\}$ for every C^P , $p = a, b$. This party constitution is easily checked to discount extremists and to be strongly symmetric. By

Theorem 11, any $2K$ -tuple (C^{a*}, C^{b*}) such that $\text{median}\{y_{a1}^*, \dots, y_{aK}^*\} = \text{median}\{y_{b1}^*, \dots, y_{bK}^*\} = \mu^*$ is a Type I party equilibrium under (e_C^K, λ_C^P) . But this case is special.

In addition to assumptions on party constitutions, the assumption that the legislative outcome function is λ_C^P also imposes structure on party organization. As remarked earlier, under λ_C^P elected party members are implicitly assumed to vote as a bloc in the legislature. In many circumstances, this is a reasonable assumption. However, even if the only two feasible policy positions that could be implemented by an elected legislature are given by the party positions, $\{g_a(\cdot), g_b(\cdot)\}$, it does not follow that any one party can guarantee all of its elected candidates will support the party platform. For instance, some successful candidate may have adopted an electoral platform y_{pr} which is some way from the party position, $g_p(\cdot)$. In this case, the candidate might abstain or even "cross the floor" and vote with the opposition. Theorem 11 can be generalized along these lines to apply to a somewhat wider class of party structures.

Let $S_p(C, \sigma)$ be the set of elected candidates who support the policy $g_p(C^P)$, given the total list of candidate positions, C , and voters' electoral strategies, σ . Define the legislative outcome function,

$$\lambda_C^Q(\cdot, \sigma) = \{y_p \in \{g_a(\cdot), g_b(\cdot)\} \mid S_p(\cdot, \sigma) > S_{p'}(\cdot, \sigma)\}.$$

So λ_C^Q differs from λ_C^P only in that not all legislative party members necessarily vote for their party's platform in the legislature. In principle, recognizing that candidates may not support the party line once elected makes the voters' decisions that much more complex, which in turn complicates the candidates' strategic calculus. In some circumstances, however, these complications are inconsequential.

Theorem 12 [Austen-Smith, 1987a]: For every candidate pr , suppose that, conditional on being elected and conditional on voting at all in the legislature, the probability that pr will support $g_p(\cdot)$ is at least as great as the probability that his electoral opponent, $p'r$, will support $g_p(\cdot)$. Suppose also

that the probability that p_r will abstain in the legislature, conditional on being elected, is the same as the probability that p_r abstains. Then (C^{a*}, C^{b*}) is a Type I party equilibrium under $(\epsilon_C^K, \lambda_C^Q)$ iff (C^{a*}, C^{b*}) is a Type I party equilibrium under $(\epsilon_C^K, \lambda_C^P)$.

The first supposition on the likely legislative behaviour of candidates is weak: it says that a given party member is more likely to support her own party than her electoral opponent is likely to support that same party. The second supposition is rather stronger, however.

The existence of two-party competition in multi-district legislative elections is a useful datum on which to support a nonstrategic view of legislative coalition-formation: there can be at most two coalitions, each one comprising the elected members of a given party. While this is theoretically convenient, and not a bad approximation to empirical reality in some polities (e.g. Great Britain), it is unsatisfactory. Political parties are themselves coalitions, and a more complete theory will explain how they come to be and to develop the forms that they do (cf. Riker, 1982). So while the party constitution approach may be a reasonable first cut at modelling multi-candidate parties in legislative elections, it is only a first cut. At the very least, the results above make clear exactly how important party structure is to understanding electoral competition with many districts.

This last observation is underscored in Austen-Smith (1986). In this paper, the election rule is still ϵ_C^K , but there are no parties: all candidates are independents. The legislature is implicitly modelled as a bargaining game in which the set of winning coalitions is $D(\cdot, \cdot)$: call this the K-legislative bargaining game. An equilibrium to the game then consists of a winning coalition, $D \in D(\cdot, \cdot)$ and the policy y_D that coalition D implements. So the legislative outcome function is,

$$\lambda_C^O(\cdot, \sigma) = \{y \in X \mid y \text{ is a component of some K-legislative bargaining equilibrium}\}.$$

Unlike in Austen-Smith and Banks (1987), the structure of the K-legislative bargaining game being played is left unspecified. Instead, voters are presumed to form (rational) beliefs about which legislature will arise, and what the subsequent K-legislative bargaining equilibrium will be.

Although this approach can be rationalized as a reduced form, it is basically ad hoc.

When deciding how to vote in this model, voters are implicitly choosing between gambles: the expected final outcome can differ in equilibrium across candidates within a given district. The convergence in (party) policies -- and hence final legislative outcomes -- associated with the deterministic models discussed above is not guaranteed here. Moreover, neither is the existence of a (Nash) equilibrium in candidate strategies. Taking Theorems 9, 11, and 12 together, it is clear that as the restrictions on party organization are relaxed, and candidates are given increasing degrees of freedom, insuring the existence of equilibrium sets of candidate platforms becomes increasingly difficult. With the total absence of parties, it is therefore not too surprising that an existence theorem is elusive (even under an assumption of quadratic utilities).

6. Conclusion

Representative democracy is inherently a multi-stage decision process. At a first stage, the electorate votes some set of candidates to legislative office; at a second stage, these agents select policy. As I argued in the Introduction, the canonic model of two candidates contesting a single district for monopolistic control of the legislature involves an essentially degenerate second stage. Since at most one candidate gets elected and that candidate simply implements his platform (whether it is known surely by the electorate, or expectationally), voters' strategic decisions are straightforward for any pair of candidate positions: vote sincerely. Consequently, ceteris paribus, candidates' strategic choices are likewise straightforward. When the legislature consists of more than a single agent, things are not so straightforward. In particular, it is only in polities for which the two-stage process is policy-irrelevant (i.e. equivalent to one-stage direct democracy) that sincere voting constitutes rational behaviour (Theorem 2). In constructing models of legislative elections, therefore, it is crucial to specify the legislative decision-making mechanism explicitly: without this, it is logically impossible to derive rational voters' strategies when confronted with some set of

electoral candidates. And without a well-defined voting equilibrium for any list of candidate platforms, candidates' strategies cannot be deduced.

Our theoretical understanding of legislative elections is as yet rudimentary. The literature is small and falls into one or other of two classes. In the first class, the legislature is elected from a national constituency by some form of proportional representation with several parties, and the final outcome is generated through legislative voting or bargaining. Unfortunately, the legislative decision-making mechanism is rarely specified and voters at the electoral stage are supposed to vote sincerely over candidates. In the second class of model, the legislature comprises the winning agents from each of K constituencies (districts), and the within-district electoral scheme is a two-candidate, simple-plurality, contest. Although the legislative decision-making mechanism is typically specified here, it is of a particularly restrictive form -- by exploiting an observation that candidates in such polities generally belong to one or other of two parties, issues of legislative coalition forming are side-stepped by presuming coalitions coincide with party members who all act as a bloc once in the assembly. Moreover, party positions (as determined from candidate platforms) are taken to be well-defined and fixed when voters go to the ballot box.

From the above perspective, it is apparent that to make progress with either class of model requires a deeper understanding of the legislative stage of the decision-making process.

Among the issues not discussed in this review (largely because there is virtually no treatment of them in the relevant literature), three are worth mentioning briefly here: candidate entry, incomplete and imperfect information, and multi-dimensional issue spaces.

The entry issue is important because the number of candidates is clearly endogenous to the legislative election scheme in operation (and not just the election rule).²¹ Whether or not candidates choose to enter an election is a positive question, the answer to which will depend on final (equilibrium) payoffs to the relevant agents (voters and candidates). Although there does exist some formal work on (loosely speaking) Duverger's Law (Palfrey, 1984), the only explicit attempt to

examine the entry question in a legislative election framework is due to Greenberg and Shepsle (1987). By construction, their concept of an $fn(K)$ -equilibrium is intended as a notion of "entry stability". In such equilibria it is not in the interest of any potential candidate to enter the election, and no extant candidate would (ceteris paribus) choose to leave the election (since she is guaranteed electoral success under the assumption of sincere voting). However, as I argued in section 4, the sincere voting assumption, coupled with a complete absence of any legislative decision-making mechanism, makes the Greenberg/Shepsle model fundamentally normative. To interpret their (non)existence result (Theorem 4) as a positive statement on candidate entry under the fixed-number election rule is, I believe, inappropriate.

All of the models discussed earlier are models of complete information. Although complete information is a natural starting point, the existence of a multi-stage representative system may ultimately rest on the presence of incomplete and imperfect information. Legislative decisions are myriad. Policy issues arise more-or-less continuously, without being fully anticipated, and are not determined in any once-and-for-all legislative decision. Moreover, legislation is rarely an end in itself, but more often an attempt to influence final consequences. Exactly how legislation translates into consequences is subject to uncertainty. Electing a representative assembly, then, is one means by which authority for handling this uncertain stream of uncertain issues is delegated from the electorate at large to a subset of that electorate.²² In a similar vein, at the electoral stage of the process, voters rarely know for sure what are the legislative implications of electing one candidate over another. In this context, the informational role of party labels in national elections is well-recognized, if not well-understood.

Finally, one-dimensional issue spaces are special. In the canonic model, assuming a single issue generates stability which typically is lost in moving to higher dimensional spaces. The relevance of incorporating a multi-dimensional issue space is evident. Unfortunately, even with a single issue, stability is hard to insure for legislative elections.

Footnotes

1. By "sensible" is meant that no individual uses a weakly dominated strategy in equilibrium.
2. This and the next section draw heavily on Austen-Smith (1987b).
3. The restriction to Euclidean preferences is in general strong. It is only for the arguments of this section that it is a convenience and not fundamental.
4. Alternatively, the example goes through (*mutatis mutandis*) if the legislative weights are given by the vote-shares relative to the total votes of the elected candidates only. See Ursprung (1980) for a similar example.
5. In particular, Greenberg's and Shepsle's Remarks 3, 5 and 6: two of these are taken up when appropriate later.
6. For instance, with sincere voting, no Nash equilibrium in candidate strategies exists if candidates attempt to vote-maximize under the fixed-standard or fixed-number rule.
7. Greenberg and Weber argue that if the legislative outcome function were given by simple majority voting over $co(C^*)$, then sincere voting is rational behaviour. The example shows this to be incorrect: any alternative y lying between $1/5$ and $4/5$ is a median outcome, and thus a legitimate policy choice for the Greenberg/Weber legislature. But only $y^* = 5/7$ insures sincere voting.
8. So there is no abstention in the model. Also, voters are permitted to use mixed strategies; but this plays no role in equilibrium.
9. As will become apparent below, if $|C^*| \leq 2$, this criterion is sufficient but not necessary for monopolistic legislative control.
10. *Parliaments of the World*, vol.II, (1986): Table 39.
11. Strictly speaking, a party's strategy also involves an acceptance rule, describing when to agree to any proposal which includes the party as a member of the government.
12. Recall that in cases of equal vote-shares, a fair random device breaks the tie before the bargaining game begins.
13. Sugden (1984), when discussing STV and not the cooperative game, allows for voters to record their entire preference ordering over a slate of candidates.
14. Approval voting is considered to be "sincere", if whenever you vote for some candidate k , you also vote for all candidates that you prefer to k .
15. The question of candidate entry into an election is considered briefly in the concluding section.
16. As usual, if vote-totals tie, a fair coin is flipped to determine the winner.
17. There does exist a theoretical literature on apportionment, however. But the analysis is

exclusively axiomatic, and not strategic. See, for instance, Balinski and Young (1982).

18. An earlier paper, Austen-Smith (1981), develops a multi-district model with many issues. To insure existence of an equilibrium (in the language developed below, a Downsian equilibrium), voting was assumed to be probabilistic in the sense of Hinich, Ledyard and Ordeshook (1972). As such, the structure is quite different from that of the rest of this essay. Moreover, probabilistic voting is essentially ad hoc. (And, for the enthusiast, be warned that the existence theorem in Austen-Smith (1981) is correct only if "quasi-concave" is strengthened to "concave".)
19. Hinich and Ordeshook (1974) examine a spatial model of the USA electoral college. The candidates in their model are isomorphic to Downsian parties, and if it is assumed that all the colleges are of the same size, then Theorem 9 is a corollary to their result.
20. Robertson is not only concerned with the impact of many districts on the Downsian model. He devotes a great deal of time to introducing party activism, campaign advertising, and so forth.
21. Consequently, analyses of the comparative properties of various election rules which assume the number of candidates is fixed are not very useful for constitutional design.
22. See Gilligan and Krehbiel (1987) for a model which suggests the importance and value of a committee structure to a larger assembly when there is imperfect information about the consequences of legislation.

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